

Non-existence of the nontrivial solution for a Sobolev type evolution inequality with nonlinear convolution term*

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Abstract: An evolution inequality of Sobolev type involving a nonlinear convolution term is considered. By using the nonlinear capacity method and the contradiction argument, the non-existence of the nontrivial local weak solution is proved.

Key words: Sobolev type evolution inequality; nonlinear convolution term; nontrivial solution; non-existence

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In this article, our attention is mainly focused on the non-existence of the nontrivial local weak solution of the following Sobolev type evolution inequality with a nonlinear convolution term

$$\begin{cases} -\partial_t \Delta u - \Delta u - \partial_t u \geq |x|^{\sigma-N} * |u|^p, & t > 0, x \in \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1)$$

where $p > 1$, $0 < \sigma < N$ and

$$|x|^{\sigma-N} * |u|^p = \int_{\mathbb{R}^N} |x-y|^{\sigma-N} |u(t, y)|^p dy.$$

Partial differential equations and inequalities with nonlinear convolution terms can be used to describe many real physical phenomena. For example, Pekar (1954) proposed

$$-\Delta u + Vu = u \left(\int_{\mathbb{R}^3} |x-y|^{-1} u^2(y) dy \right), \quad x \in \mathbb{R}^3 \quad (2)$$

and used it to model the quantum mechanics of a polaron at rest. (2) can also be used to model an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma (one can see Lieb (1977) for more details). Over past years, many mathematicians have paid attention to the existence and non-existence of the solutions to various nonlinear parabolic problems with nonlocal nonlinearities (see Duruk et al., 2010; Moroz et al., 2013, 2017; Ghergu et al., 2016; Ikeda et al., 2020; Filippucci et al., 2022a and the references therein). For instance, by using the nonlinear capacity method, Mitidieri et al. (2005) dealt with the following parabolic inequality

$$\partial_t u - \Delta u \geq \int_{\mathbb{R}^N} |x-y|^{\beta-N} |u(t, y)|^q dy, \quad t > 0, x \in \mathbb{R}^N, \quad (3)$$

where $0 < \beta < N$ and $q > 1$. They proved that if

$$u(0, \cdot) \in L^1_{loc}(\mathbb{R}^N), \quad 1 < q \leq \frac{N+2}{N-\beta} \quad \text{and} \quad \liminf_{R \rightarrow \infty} R^{-\beta} \int_{|x| < R} u(0, x) dx \geq 0,$$

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then problem (3) admits no nontrivial weak solution. Recently, Filippucci et al. (2022b) investigated the following higher order evolution inequalities

$$\begin{cases} \partial_t^k u + (-\Delta)^m u \geq |u|^q \left(\int_{\mathbb{R}^N} K(x-y) |u(t,y)|^p dy \right), & t > 0, x \in \mathbb{R}^N, \\ \partial_t^i u(0, x) = u_i(x), & i = 0, 1, \dots, k-1, x \in \mathbb{R}^N, \end{cases} \quad (4)$$

where $\partial_t^i = \frac{\partial^i}{\partial t^i}$ and $N, k, m \geq 1$ are positive integers, $p > 0, q > 0$ and the kernel $K \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $K(|x|) \in L^1_{loc}(\mathbb{R}^N)$ and $\inf_{0 < r < R} K(r) = K(R)$ for sufficiently large R . The authors gave the sufficient conditions on the existence and non-existence of the nontrivial solution of problem (4).

On the other hand, there are a lot of works on the instantaneous blow-up for evolution equations and inequalities of the Sobolev type. Coleman et al. (1965) studied the problem

$$\begin{cases} \partial_t(\partial_{xx}u + u) = \partial_{xx}u, & t > 0, x \in (0, L), \\ u(0, x) = u_0(x), & x \in (0, L), \\ u(t, 0) = u(t, L), & t > 0. \end{cases} \quad (5)$$

The authors concluded that (5) possesses no bounded solution on an arbitrarily small interval of time provided that $0 < L \leq \pi$.

Korpusov (2015) handled the instantaneous blow-up of the solution to the following Sobolev type inequality

$$\begin{cases} -\partial_t \Delta u - \Delta u \geq |u|^p, & t > 0, x \in \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (6)$$

where $p > 1$. The author showed that the critical exponent of instantaneous blow-up for problem (6) is

$$p_* = \begin{cases} \frac{N}{N-2}, & \text{if } N \geq 3, \\ +\infty, & \text{if } N \in \{1, 2\}. \end{cases}$$

Specifically speaking, if $1 < p \leq p_*$, then problem (6) has no nontrivial local weak solution under some certain conditions on the initial datum; if $p > p_*$, then problem (6) possesses nontrivial local weak solution for some $u_0(x)$.

Very recently, Alazman et al. (2023) dealt with the following evolution inequality of Sobolev type involving nonlocal nonlinearity

$$\begin{cases} -\partial_t \Delta u - \Delta u \geq \int_{\mathbb{R}^N} |x-y|^{\sigma-N} |u(t,y)|^p dy, & t > 0, x \in \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (7)$$

where $0 < \sigma < N$ and $p > 1$. The critical exponent of instantaneous blow-up to problem (7) is given by

$$p_*(\sigma, N) = \begin{cases} \frac{N}{N-2-\sigma}, & \text{if } \sigma < N-2, \\ +\infty, & \text{if } \sigma \geq N-2. \end{cases}$$

For other work related to the instantaneous blow-up to the problems of Sobolev type, we refer readers to (Korpusov, 2015; Jleli et al., 2020, 2022) and the references therein. As far as we know, there are no results on the existence and non-existence of the nontrivial weak solution to problem (1). It is interesting to evaluate the effect of the nonlinear convolution term on the existence of the nontrivial weak solution to problem (1). Motivated by the above works, our main goal of this article is to give an answer on the non-existence of the nontrivial weak solution to problem (1). The definition of the weak solution of problem (1) is given as follows.

Definition 1 A function $u(t, x)$ is said to be a local weak solution to problem (1) if there exists $T \in (0, +\infty)$ such that $u(t, x) \in L^p_{loc}([0, T] \times \mathbb{R}^N)$, $u(0, x) = u_0(x) \in L^1_{loc}(\mathbb{R}^N)$, the function

$$(t, x) \mapsto \int_{\mathbb{R}^N} |x-y|^{\sigma-N} |u(t,y)|^p dy$$

belongs to $L^1_{loc}([0, T] \times \mathbb{R}^N)$, and the integral inequality

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\sigma - N} |u(t, y)|^p \varphi(t, x) dx dy dt \\ & \leq \int_0^T \int_{\mathbb{R}^N} u(\Delta\varphi_t - \Delta\varphi + \varphi_t) dx dt + \int_{\mathbb{R}^N} u_0(x)(\Delta\varphi(0, x) + \varphi(0, x)) dx \end{aligned} \quad (8)$$

holds for any test function

$$\varphi(t, x) \in C^3([0, T] \times \mathbb{R}^N) \text{ with } \varphi(t, x) \geq 0, \varphi(T, x) \equiv 0, \text{ and } \text{supp}_x \varphi(t, x) \subset\subset \mathbb{R}^N.$$

The main result of the present article is the following theorem.

Theorem 1 Suppose that $u_0 \in L^1(\mathbb{R}^N)$, $0 < \sigma < N$ and $1 < p \leq \frac{N}{N - \sigma}$. Then (1) admits no nontrivial local weak solution.

1 A priori estimates

In this section, we will give some a priori estimates as preliminaries, which play a crucial role later.

Lemma 1 Let $\eta(t) \in C^3([0, T])$ with $\eta(t) \geq 0$ and $\eta(T) = 0$, $\chi(x) \in C^3(\mathbb{R}^N)$ satisfying $\chi(x) \geq 0$ and $\text{supp} \chi(x) \subset\subset \mathbb{R}^N$. Suppose that $u(t, x) \in L^p_{loc}([0, T] \times \mathbb{R}^N)$ is a local weak solution to (1). Then, one has

$$\int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\sigma - N} |u(t, y)|^p \eta(t) \chi(x) dx dy dt \leq \frac{p\eta(0)}{p - 1} \int_{\mathbb{R}^N} u_0(x)(\Delta\chi(x) + \chi(x)) dx + 2^{\frac{1}{p-1}} I(\eta, \chi), \quad (9)$$

where

$$I(\eta, \chi) = \left(\int_{\mathbb{R}^N} \chi(x)^{\frac{p+1}{p}} dx \right)^{-\frac{p}{p-1}} \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\frac{N-\sigma}{p-1}} \eta(t)^{-\frac{1}{p-1}} \chi(y)^{\frac{p}{p-1}} J(\eta, \chi) dx dy dt, \quad (10)$$

$$J(\eta, \chi) = |\eta'(t) \Delta\chi(x)|^{\frac{p}{p-1}} + (\eta(t) |\Delta\chi(x)|)^{\frac{p}{p-1}} + (|\eta'(t) \chi(x)|)^{\frac{p}{p-1}}. \quad (11)$$

Proof Let $\varphi(t, x) \in C^3([0, T] \times \mathbb{R}^N)$ be a function fulfilling

$$\varphi(t, x) \geq 0, \text{ supp}_x \varphi(t, x) \subset\subset \mathbb{R}^N \text{ and } \varphi(T, x) \equiv 0.$$

Then by (8), one has

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\sigma - N} |u(t, y)|^p \varphi(t, x) dx dy dt \\ & \leq \int_0^T \int_{\mathbb{R}^N} u(t, x)(\mathcal{L}\varphi)(t, x) dx dt + \int_{\mathbb{R}^N} u_0(x)(\Delta\varphi(0, x) + \varphi(0, x)) dx, \end{aligned} \quad (12)$$

where

$$\mathcal{L}\varphi = \Delta\varphi_t - \Delta\varphi + \varphi_t. \quad (13)$$

Let $\vartheta(x) \in C(\mathbb{R}^N)$ be a nonnegative compactly supported function. Then

$$\begin{aligned} & \int_{\mathbb{R}^N} \vartheta(y) dy \int_0^T \int_{\mathbb{R}^N} u(t, x)(\mathcal{L}\varphi)(t, x) dx dt \\ & = \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u(t, x)(\mathcal{L}\varphi)(t, x) \vartheta(y) dx dy dt \\ & = \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(|x - y|^{\frac{\sigma - N}{p}} u(t, x) \varphi(t, y)^{\frac{1}{p}} \right) \cdot \left(|x - y|^{\frac{N - \sigma}{p}} (\mathcal{L}\varphi)(t, x) \varphi(t, y)^{-\frac{1}{p}} \vartheta(y) \right) dx dy dt. \end{aligned}$$

With the help of ε -Young inequality with $\varepsilon = \frac{1}{p} \int_{\mathbb{R}^N} \vartheta(y) dy$, one can conclude that

$$\begin{aligned} & \int_{\mathbb{R}^N} \vartheta(y) dy \int_0^T \int_{\mathbb{R}^N} u(t, x)(\mathcal{L}\varphi)(t, x) dx dt \\ & \leq \frac{1}{p} \int_{\mathbb{R}^N} \vartheta(y) dy \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\sigma - N} |u(t, x)|^p \varphi(t, y) dx dy dt \\ & \quad + \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\frac{N - \sigma}{p-1}} |(\mathcal{L}\varphi)(t, x)|^{\frac{p}{p-1}} \varphi(t, y)^{-\frac{1}{p-1}} \vartheta(y)^{\frac{p}{p-1}} dx dy dt \cdot \frac{p-1}{p} \left(\int_{\mathbb{R}^N} \vartheta(y) dy \right)^{-\frac{1}{p-1}}. \end{aligned}$$

Combining the above inequality with (12) yields that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\sigma - N} |u(t, y)|^p \varphi(t, x) dx dy dt \\ & \leq \left(\int_{\mathbb{R}^N} \vartheta(y) dy \right)^{-\frac{p}{p-1}} \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\frac{N - \sigma}{p-1}} |(\mathcal{L}\varphi)(t, x)|^{\frac{p}{p-1}} \varphi(t, y)^{-\frac{1}{p-1}} \vartheta(y)^{\frac{p}{p-1}} dx dy dt \\ & \quad + \frac{p}{p-1} \int_{\mathbb{R}^N} u_0(x) (\Delta\varphi(0, x) + \varphi(0, x)) dx. \end{aligned} \tag{14}$$

Now, taking $\varphi(t, x) = \eta(t)\chi(x)$ and $\vartheta(y) = \chi(y)^{\frac{p+1}{p}}$, then from (14) and the fact

$$|\Delta\varphi_t - \Delta\varphi + \varphi_t|^{\frac{p}{p-1}} \leq |\Delta\varphi_t|^{\frac{p}{p-1}} + |\Delta\varphi|^{\frac{p}{p-1}} + |\varphi_t|^{\frac{p}{p-1}},$$

one can claim that (9) holds. The proof of Lemma 1 is complete.

Now, let $\xi \in C^\infty([0, +\infty))$ be a cut-off function with $0 \leq \xi \leq 1$,

$$\xi(s) = 1 \text{ for } 0 \leq s \leq \frac{1}{2}, \text{ and } \xi(s) = 0 \text{ for } s \geq 1. \tag{15}$$

For given $R > 0$ and sufficiently large ℓ (Mitidieri et al., 2005), define

$$\eta(t) = \xi\left(\frac{t}{T}\right)^\ell, \quad 0 \leq t \leq T, \tag{16}$$

and

$$\chi(x) = R^{-\sigma} \xi\left(\frac{|x|}{R}\right), \quad x \in \mathbb{R}^N. \tag{17}$$

After a series of simple calculations, one knows that

$$\int_{\mathbb{R}^N} \chi(x)^{\frac{p+1}{p}} dx = \int_{\mathbb{R}^N} \left(R^{-\sigma} \xi\left(\frac{|x|}{R}\right) \right)^{\frac{p+1}{p}} dx = R^{-\frac{\sigma(p+1)}{p}} \int_{\mathbb{R}^N} \xi\left(\frac{|x|}{R}\right)^{\frac{p+1}{p}} dx \geq R^{-\frac{\sigma(p+1)}{p}} \int_{|x| \leq \frac{R}{2}} dx \geq CR^{N - \frac{\sigma(p+1)}{p}},$$

which means that

$$\left(\int_{\mathbb{R}^N} \chi(x)^{\frac{p+1}{p}} dx \right)^{-\frac{p}{p-1}} \leq CR^{\frac{\sigma(p+1) - Np}{p-1}}. \tag{18}$$

Meanwhile, one has

$$\int_0^T \eta(t) dt = \int_0^T \xi\left(\frac{t}{T}\right)^\ell dt = T \int_0^1 \xi(s)^\ell ds = CT,$$

and

$$\begin{aligned} \int_0^T \eta(t)^{-\frac{1}{p-1}} |\eta'(t)|^{\frac{p}{p-1}} dt &= \ell^{\frac{p-1}{p}} T^{-\frac{p}{p-1}} \int_0^T \xi\left(\frac{t}{T}\right)^{\ell - \frac{p}{p-1}} \left| \xi'\left(\frac{t}{T}\right) \right|^{\frac{p}{p-1}} dt \\ &= \ell^{\frac{p-1}{p}} T^{-\frac{1}{p-1}} \int_0^1 \xi(s)^{\ell - \frac{p}{p-1}} |\xi'(s)|^{\frac{p}{p-1}} ds = CT^{-\frac{1}{p-1}}. \end{aligned}$$

Lemma 2 One has

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\frac{N - \sigma}{p-1}} \eta(t)^{-\frac{1}{p-1}} \chi(y)^{\frac{p}{p-1}} J(\eta, \chi) dx dy dt \\ & \leq CT^{-\frac{1}{p-1}} \left(R^{2N + \frac{N - \sigma - 2p(\sigma + 1)}{p-1}} + R^{2N + \frac{N - \sigma - 2\sigma p}{p-1}} \right) + CTR^{2N + \frac{N - \sigma - 2p(\sigma + 1)}{p-1}}, \end{aligned} \tag{19}$$

and

$$I(\eta, \chi) \leq CT^{-\frac{1}{p-1}} \left(R^{N - \frac{p(\sigma + 2)}{p-1}} + R^{N - \frac{\sigma p}{p-1}} \right) + CTR^{N - \frac{p(\sigma + 2)}{p-1}}, \tag{20}$$

where $I(\eta, \chi)$, $J(\eta, \chi)$, $\eta(t)$ and $\chi(x)$ are defined by (10), (11), (16) and (17), respectively.

Proof First, recalling that $N > \sigma$ and $p > 1$, one has

$$|x - y|^{\frac{N-\sigma}{p-1}} \leq C \left(|x|^{\frac{N-\sigma}{p-1}} + |y|^{\frac{N-\sigma}{p-1}} \right),$$

which leads us to

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\frac{N-\sigma}{p-1}} (\chi(y) |\eta'(t) \Delta \chi(x)|)^{\frac{p}{p-1}} \eta(t)^{-\frac{1}{p-1}} dx dy dt \\ & \leq CT^{-\frac{1}{p-1}} \left(\int_{\mathbb{R}^N} |x|^{\frac{N-\sigma}{p-1}} |\Delta \chi(x)|^{\frac{p}{p-1}} dx \right) \left(\int_{\mathbb{R}^N} \chi(y)^{\frac{p}{p-1}} dy \right) \\ & \quad + CT^{-\frac{1}{p-1}} \left(\int_{\mathbb{R}^N} |\Delta \chi(x)|^{\frac{p}{p-1}} dx \right) \left(\int_{\mathbb{R}^N} |y|^{\frac{N-\sigma}{p-1}} \chi(y)^{\frac{p}{p-1}} dy \right). \end{aligned} \quad (21)$$

On the other hand, from (15) and (17), it follows that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{\frac{N-\sigma}{p-1}} |\Delta \chi(x)|^{\frac{p}{p-1}} dx & = R^{-\frac{\sigma p}{p-1}} \int_{\frac{R}{2} < |x| < R} |x|^{\frac{N-\sigma}{p-1}} \left| \Delta \left[\xi \left(\frac{|x|}{R} \right) \right] \right|^{\frac{p}{p-1}} dx \\ & \leq CR^{-\frac{p(\sigma+2)}{p-1}} \int_{\frac{R}{2} < |x| < R} |x|^{\frac{N-\sigma}{p-1}} dx \leq CR^{N + \frac{N-\sigma-p(\sigma+2)}{p-1}}, \end{aligned} \quad (22)$$

and

$$\int_{\mathbb{R}^N} |\Delta \chi(x)|^{\frac{p}{p-1}} dx \leq R^{-\frac{p(\sigma+2)}{p-1}} \int_{\mathbb{R}^N} dx \leq CR^{N - \frac{p(\sigma+2)}{p-1}}. \quad (23)$$

Using (15) and (17) again yields that

$$\int_{\mathbb{R}^N} |y|^{\frac{N-\sigma}{p-1}} \chi(y)^{\frac{p}{p-1}} dy = \int_{|y| \leq R} |y|^{\frac{N-\sigma}{p-1}} \left(R^{-\sigma} \xi \left(\frac{|y|}{R} \right) \right)^{\frac{p}{p-1}} dy \leq R^{-\frac{\sigma p}{p-1}} \int_{|y| \leq R} |y|^{\frac{N-\sigma}{p-1}} dy \leq CR^{\frac{N-\sigma(p+1)}{p-1} + N}, \quad (24)$$

and

$$\int_{\mathbb{R}^N} \chi(y)^{\frac{p}{p-1}} dy = \int_{|y| \leq R} \left(R^{-\sigma} \xi \left(\frac{|y|}{R} \right) \right)^{\frac{p}{p-1}} dy \leq R^{-\frac{\sigma p}{p-1}} \int_{|y| \leq R} dy \leq CR^{N - \frac{\sigma p}{p-1}}. \quad (25)$$

Summing up (21)-(25), one immediately arrives at

$$\int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\frac{N-\sigma}{p-1}} (\chi(y) |\eta'(t) \Delta \chi(x)|)^{\frac{p}{p-1}} \eta(t)^{-\frac{1}{p-1}} dx dy dt \leq CT^{-\frac{1}{p-1}} R^{2N + \frac{N-\sigma-2p(\sigma+1)}{p-1}}. \quad (26)$$

On the other hand, one has

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\frac{N-\sigma}{p-1}} \eta(t)^{-\frac{1}{p-1}} \chi(y)^{\frac{p}{p-1}} (\eta(t) |\Delta \chi(x)|)^{\frac{p}{p-1}} dx dy dt \\ & \leq CT \left(\int_{\mathbb{R}^N} |x|^{\frac{N-\sigma}{p-1}} |\Delta \chi(x)|^{\frac{p}{p-1}} dx \right) \left(\int_{\mathbb{R}^N} \chi(y)^{\frac{p}{p-1}} dy \right) + CT \left(\int_{\mathbb{R}^N} |\Delta \chi(x)|^{\frac{p}{p-1}} dx \right) \left(\int_{\mathbb{R}^N} |y|^{\frac{N-\sigma}{p-1}} \chi(y)^{\frac{p}{p-1}} dy \right) \\ & \leq CTR^{2N + \frac{N-\sigma-2p(\sigma+1)}{p-1}}. \end{aligned} \quad (27)$$

Analogously, one can deduce that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\frac{N-\sigma}{p-1}} \eta(t)^{-\frac{1}{p-1}} \chi(y)^{\frac{p}{p-1}} (|\eta'(t) \chi(x)|)^{\frac{p}{p-1}} dx dy dt \\ & \leq CT^{-\frac{1}{p-1}} \left(\int_{\mathbb{R}^N} |x|^{\frac{N-\sigma}{p-1}} \chi(x)^{\frac{p}{p-1}} dx \right) \left(\int_{\mathbb{R}^N} \chi(y)^{\frac{p}{p-1}} dy \right) + CT^{-\frac{1}{p-1}} \left(\int_{\mathbb{R}^N} \chi(x)^{\frac{p}{p-1}} dx \right) \left(\int_{\mathbb{R}^N} |y|^{\frac{N-\sigma}{p-1}} \chi(y)^{\frac{p}{p-1}} dy \right) \\ & \leq CT^{-\frac{1}{p-1}} R^{2N + \frac{N-\sigma-2\sigma p}{p-1}}. \end{aligned} \quad (28)$$

Combining (26), (27) and (28) yields the desired estimate (19). Finally, inserting (18) and (19) into (10), one can find that (20) holds. The proof of Lemma 2 is complete.

Lemma 3 Let

$$\varphi(t, x) = \eta(t)\chi(x), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^N.$$

Then one has

$$\left(\int_0^T \int_{\frac{R}{2} < |x| < R} |(\mathcal{L}\varphi)(t, x)|^{\frac{p}{p-1}} \eta(t)^{-\frac{1}{p-1}} dx dt \right)^{\frac{p-1}{p}} \leq CT^{\frac{p-1}{p}} R^{\frac{N(p-1)}{p} - \sigma}, \quad (29)$$

where $\mathcal{L}\varphi$, $\eta(t)$ and $\chi(x)$ are given by (13), (16) and (17), respectively.

Proof By (13), (15), (16) and (17), one can conclude that, for $0 \leq t \leq T$, $\frac{R}{2} < |x| < R$,

$$\begin{aligned} |(\mathcal{L}\varphi)(t, x)|^{\frac{p}{p-1}} \eta(t)^{-\frac{1}{p-1}} &= |\Delta\varphi_t - \Delta\varphi + \varphi_t|^{\frac{p}{p-1}} \xi\left(\frac{t}{T}\right)^{-\frac{\ell}{p-1}} \\ &= |\eta'(t)\Delta\chi(x) - \eta(t)\Delta\chi(x) + \eta'(t)\chi(x)|^{\frac{p}{p-1}} \xi\left(\frac{t}{T}\right)^{-\frac{\ell}{p-1}} \\ &\leq C\xi\left(\frac{t}{T}\right)^{\ell - \frac{p}{p-1}} \left(|\Delta\chi(x)|^{\frac{p}{p-1}} + \chi(x)^{\frac{p}{p-1}} \right) \leq C \left(R^{-\frac{p(\sigma+2)}{p-1}} + R^{\frac{-\sigma p}{p-1}} \right) \leq CR^{-\frac{\sigma p}{p-1}}. \end{aligned}$$

Then

$$\int_0^T \int_{\frac{R}{2} < |x| < R} |(\mathcal{L}\varphi)(t, x)|^{\frac{p}{p-1}} \eta(t)^{-\frac{1}{p-1}} dx dt \leq CTR^{N - \frac{\sigma p}{p-1}},$$

which equals to (29). The proof of Lemma 3 is complete.

Lemma 4 Let $u(t, x) \in L^p_{loc}([0, T] \times \mathbb{R}^N)$ be a local weak solution to (1). Then

$$\int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\sigma - N} |u(t, y)|^p \eta(t)\chi(x) dx dy dt \geq C \int_0^T \int_{|x| < \frac{R}{2}} \xi\left(\frac{t}{T}\right)^\ell |u(t, x)|^p dx dt,$$

where $\eta(t)$ and $\chi(x)$ are given by (16) and (17), respectively.

Proof Using the inequality

$$|x - y|^{\sigma - N} \geq (|x| + |y|)^{\sigma - N}$$

for $\sigma \in (0, N)$, one sees that

$$\int_{\mathbb{R}^N} |x - y|^{\sigma - N} \chi(x) dx \geq \int_{\mathbb{R}^N} (|x| + |y|)^{\sigma - N} \chi(x) dx \geq R^{-\sigma} \int_{|x| < \frac{R}{2}} (|x| + |y|)^{\sigma - N} dx \geq C \left(\frac{R}{2} + |y| \right)^{\sigma - N} R^{N - \sigma}.$$

In light of the above inequality, one has

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\sigma - N} |u(t, y)|^p \eta(t)\chi(x) dx dy dt \\ &= \int_0^T \eta(t) \int_{\mathbb{R}^N} |u(t, y)|^p \left(\int_{\mathbb{R}^N} |x - y|^{\sigma - N} \chi(x) dx \right) dy dt \\ &\geq C \int_0^T \eta(t) \int_{\mathbb{R}^N} |u(t, y)|^p \left(\frac{R}{2} + |y| \right)^{\sigma - N} R^{N - \sigma} dy dt \\ &\geq C \int_0^T \eta(t) \int_{|y| < \frac{R}{2}} |u(t, y)|^p \left(\frac{R}{2} + |y| \right)^{\sigma - N} R^{N - \sigma} dy dt \geq C \int_0^T \int_{|y| < \frac{R}{2}} \xi\left(\frac{t}{T}\right)^\ell |u(t, y)|^p dy dt, \end{aligned}$$

which completes the proof of Lemma 4.

2 Proof of Theorem 1

Suppose by contradiction that $u(t, x) \in L^p_{loc}([0, T] \times \mathbb{R}^N)$ is a nontrivial local weak solution to problem (1).

Then a simple application of Lemma 1 yields that

$$\int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{\frac{N - \sigma}{p-1}} |u(t, y)|^p \eta(t)\chi(x) dx dy dt \leq \frac{P}{p-1} \int_{\mathbb{R}^N} u_0(x)(\Delta\chi(x) + \chi(x)) dx + 2^{\frac{1}{p-1}} I(\eta, \chi),$$

where η and χ are defined in (16) and (17), respectively. Furthermore, from (20) and Lemma 4, it follows that

$$\begin{aligned} & \int_0^T \int_{|x| < \frac{R}{2}} \xi \left(\frac{t}{T} \right) |u(t, x)|^p dx dt \\ & \leq C \left[\int_{\mathbb{R}^N} u_0(x) (\Delta \chi(x) + \chi(x)) dx + T^{-\frac{1}{p-1}} \left(R^{N - \frac{p(\sigma+2)}{p-1}} + R^{N - \frac{\sigma p}{p-1}} \right) + TR^{N - \frac{p(\sigma+2)}{p-1}} \right]. \end{aligned} \quad (30)$$

On the other hand, combining (15) with (17), one has

$$\left| \int_{\mathbb{R}^N} u_0(x) (\Delta \chi(x) + \chi(x)) dx \right| \leq \int_{\frac{R}{2} < |x| < R} |u_0(x)| |\Delta \chi(x) + \chi(x)| dx \leq C(R^{-\sigma-2} + R^{-\sigma}) \int_{\frac{R}{2} < |x| < R} |u_0(x)| dx.$$

Remembering that $u_0 \in L^1(\mathbb{R}^N)$, one can deduce from the above inequality that

$$\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^N} u_0(x) (\Delta \chi(x) + \chi(x)) dx = 0. \quad (31)$$

If $0 < \sigma < N$ and $1 < p < \frac{N}{N-\sigma}$. Then $N - \frac{p(\sigma+2)}{p-1} < 0$ and $N - \frac{\sigma p}{p-1} < 0$. In this case, by passing to the limit as $R \rightarrow +\infty$ in (30) and using (31), one obtains that

$$\int_0^T \int_{\mathbb{R}^N} \xi \left(\frac{t}{T} \right) |u(t, x)|^p dx dt = 0,$$

which contradicts the fact that $u(t, x)$ is nontrivial.

If $0 < \sigma < N$ and $p = \frac{N}{N-\sigma}$. In this case, one has $N - \frac{\sigma p}{p-1} = 0$. Thus, by (30) and (31), one can deduce that

$$\int_0^T \int_{\mathbb{R}^N} \xi \left(\frac{t}{T} \right) |u(t, x)|^p dx dt < +\infty. \quad (32)$$

Let φ be a test function defined by (11). Then, by (8), one has

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{\sigma-N} |u(t, y)|^p \eta(t) \chi(x) dx dy dt \\ & \leq \int_0^T \int_{\mathbb{R}^N} u(t, x) (\mathcal{L}\varphi)(t, x) dx dt + \int_{\mathbb{R}^N} u_0(x) (\Delta \chi(x) + \chi(x)) dx, \end{aligned} \quad (33)$$

where $\mathcal{L}\varphi$ is given by (13). Keeping (15) in mind and using Hölder's inequality tell us that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} u(t, x) (\mathcal{L}\varphi)(t, x) dx dt \\ & \leq \int_0^T \int_{\frac{R}{2} < |x| < R} \xi \left(\frac{t}{T} \right)^{\frac{\ell}{p}} |u(t, x)| \xi \left(\frac{t}{T} \right)^{-\frac{\ell}{p}} |(\mathcal{L}\varphi)(t, x)| dx dt \\ & \leq \left(\int_0^T \int_{\frac{R}{2} < |x| < R} \xi \left(\frac{t}{T} \right)^{\ell} |u(t, x)|^p dx dt \right)^{\frac{1}{p}} \cdot \left(\int_0^T \int_{\frac{R}{2} < |x| < R} \xi \left(\frac{t}{T} \right)^{-\frac{\ell}{p-1}} |(\mathcal{L}\varphi)(t, x)|^{\frac{p-1}{p}} dx dt \right)^{\frac{p-1}{p}}, \end{aligned}$$

which together with Lemma 3 yield that

$$\int_0^T \int_{\mathbb{R}^N} u(t, x) (\mathcal{L}\varphi)(t, x) dx dt \leq CT^{\frac{p-1}{p}} R^{\frac{N(p-1)}{p} - \sigma} \left(\int_0^T \int_{\frac{R}{2} < |x| < R} \xi \left(\frac{t}{T} \right)^{\ell} |u(t, x)|^p dx dt \right)^{\frac{1}{p}}. \quad (34)$$

With (33) and (34) in hand and using Lemma 4, one can claim that

$$\begin{aligned} & C \int_0^T \int_{|x| < \frac{R}{2}} \xi \left(\frac{t}{T} \right)^{\ell} |u(t, x)|^p dx dt \\ & \leq CT^{\frac{p-1}{p}} R^{\frac{N(p-1)}{p} - \sigma} \left(\int_0^T \int_{\frac{R}{2} < |x| < R} \xi \left(\frac{t}{T} \right)^{\ell} |u(t, x)|^p dx dt \right)^{\frac{1}{p}} + \int_{\mathbb{R}^N} u_0(x) (\Delta \chi(x) + \chi(x)) dx. \end{aligned}$$

Passing to the limit as $R \rightarrow +\infty$ in the above inequality, recalling (31) and (32), one can conclude that

$$\int_0^T \int_{\mathbb{R}^n} \xi \left(\frac{t}{T} \right) |u(t, x)|^p dx dt = 0,$$

which contradicts the fact that $u(t, x)$ is nontrivial. This completes the proof of Theorem 1.

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具有非线性卷积项的 Sobolev 型演化不等式非平凡解的不存在性

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摘要: 研究一类具非线性卷积项的 Sobolev 型演化不等式. 结合非线性容量证法及反证法, 给出了非平凡解的不存在性结果.

关键词: Sobolev 型演化不等式; 非线性卷积项; 非平凡解; 不存在性

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